COMBINED FIRST AND SECOND SEMESTER B.TECH.(ENGINEERING) DEGREE EXAMINATION,APRIL 2015

MODEL QUESTION PAPER-I (2014 SCHEME)

EN 14 101-ENGINEERING MATHEMATICS -1

TIME: THREE HOURS **MAXIMUM:100 Marks**

PART A

ANSWER ANY EIGHT QUESTIONS

1.If z is a homogeneous function of degree n in x and y,show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial z^2}=n(n-1)z.$$

2.Test the convergence of the series $\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\cdots$

3.Verify Cayley Hamilton theorem for the matrix A= $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

4.Determine the rank of the following matrix A= $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

5.Find the Fourier series of the function $f(x)=2x-x^2$ for 0 < x < 3 and $f(x+3)=f(x)$.

6.Find the half range cosine series for the function $f(x)=x^2$ in the range 0 ≤x ≤ π

7.Evaluate $\lim_{x\to 0}\left(\frac{sinx}{x}\right)^{1/x^2}$

8.Find the radius of curvature at any point (x,y) on the parabola $y^2=4ax$

9.Test the convergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\cdots$

10.Expand $cosx$ as a Maclaurin's series containing terms up to x^4

(8×5=40 marks)

PART B

11.(a)(i) Find the evolute of the curve $x^{2/3} + y^{2/3} = a^{2/3}$. (8 marks)

(a)(ii) Examine the function $x^3 + y^3 - 3axy$ for minima and maxima (7 marks)

OR

(b)(i) Examine for minimum and maximum values of: $\sin x + \sin y + \sin x + y$ (8 marks)

(b)(ii) If Z is a function of x and y,where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$,show that

$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ (7 marks)

12 (a)(i) Test the convergence of the series $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots$ (8 marks)

(a)(ii) If $y = \frac{1}{\sqrt{1+2x}}$ .Prove that$(1 + 2x)y_{n+1} + (2n + 1)y_n = 0$ (7 marks)

OR

(b)(i) Find the interval of convergence of the series $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \cdots$ (8 marks)

(b)(ii) If $y = a\cos(logx) + b\sin(logx)$.

Prove that $x^2y_{n+1} + (2n + 1)xy_{n+1} + (n^2 + 1)y_n$=0 (7 marks)

13 (a)(i)Find the eigen values of the matrix$\left[\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}\right]$ Hence find the matrix whose

eigen values are $\frac{1}{6}$ $and - 1$ (7 marks)

(a)(ii) Solve the system of equations

$$x + 2y - z = 3,3x - y + 2z = 1,2x - 2y + 3z = 2, x - y + z = -1$$

(8 marks)

OR

(13)(b) Reduce the quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy + 4xz - 8yz\ in$ to Canonical form by orthogonal reduction.

(15 marks)

(14)(a)(i) Show that for –π <x < π

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \cdots\right]$$ (8 marks)

(a)(ii) Obtain the half range sine series for $e^x\ in\ 0 < x < 1$ (7 marks)

OR

14(b) The following values of y give the displacement in inches of a certain machine part for the rotation x of the flywheel .Expand y in the form of a Fourier series

x :	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$
y:	0	9.2	14.4	17.8	17.3	11.7

(15 marks)

(4×15=60 marks)

Scheme of Valuation of I & II Semester B.Tech Modal Question Paper-I, April 2015

EN14-101- Engineering Mathematics-I

Sl.No	Value Points / Answer	Mark
1	If z is a homogeneous function of degree 'n' then by Euler's theorem, $x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=nz \longrightarrow (1)$	1
	Differentiating (1) partially w.r.t x, we have $x.\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}+y.\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$	1
	Differentiating (1) Partially w.r.t y, we have $x.\frac{\partial^2 z}{\partial y\partial x}+1.\frac{\partial z}{\partial y}+y.\frac{\partial^2 z}{\partial y^2}=n.\frac{\partial z}{\partial y}$ (2)	1
	But $\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial^2 z}{\partial x\partial y}$	
	$\therefore x.\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial y}+y\frac{\partial^2 z}{\partial y^2}=n.\frac{\partial z}{\partial y} \longrightarrow (3)$	
	Multiplying (2) by x, (3) by y and adding,	
	$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)=n.\left(x.\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$	
	or $x^2.\frac{\partial^2 z}{\partial x^2}+2xy.\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+nz=n.nz$	
	$\Rightarrow x^2.\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2.\frac{\partial^2 z}{\partial y^2}=n^2u-nu=n(n-1)u$	2
2.	Neglecting 1st term, $u_n=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$ $\left\{\text{or } u_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}}\right\}$	1
	$u_{n+1}=\frac{x^{2n+2}}{(n+3)\sqrt{n+2}}$	
	$\frac{u_{n+1}}{u_n}=\frac{x^{2n+2}}{(n+3)\sqrt{n+2}}\times\frac{(n+2)\sqrt{n+1}}{x^{2n}}=x^2 \longrightarrow (1)$	1
	If $x^2<1$, convergent	1
	If $x^2>1$, divergent and test fails for $x^2=1$ (ie for $x=\pm1$)	
	If $x^2=1$, $u_n=\frac{1}{(n+2)\sqrt{n+1}}$, Choose $v_n=\frac{1}{n^{3/2}}$	1
	$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=1\neq0$. By Limit form of Comparison test Σu_n and Σv_n converges or diverges together.	
	Σv_n Converges $\Rightarrow \Sigma u_n$ also Converges	1
	$\therefore \Sigma u_n$ Converges for $x^2\leq1$ & diverges if $x^2>1$	

Sl. NO	Value Points / Answer	Mark
3.	Characteristic Equation $\lvert A-\lambda I\rvert = 0$	1
	$\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$	
	$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$	
	By Cayley-Hamilton theorem $A^3 - 5A^2 + 7A - 3I = 0$	1
	$A^3 = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$	
	$A^2 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$	1
	$\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 5\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 7\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	2
	$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
	Hence Cayley Hamilton theorem is verified.	
4.	$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$	
	$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $\quad R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$	1
	$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $\quad R_2 \longleftrightarrow R_3$	1
	$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & -3 & +2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$ $\quad R_2 \to \frac{R_2}{-4}$	1

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Sl.NO	Value Points / Answer 3	Mark
	$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix} \quad R_4 \to R_4 + 4R_2$	1
	$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -3/4 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$	1
	No: of non-zero rows in echelon form = rank of $A = 3$	
5.	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n x\pi}{l} + b_n \sin\frac{n\pi x}{l}\right)$ where $l = 3/2$	1
	$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$	
	$= \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (2x - x^2)\,dx = \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$	
	$= \frac{2}{3}\left[9 - \frac{27}{3}\right] = 0$	1
	$a_n = \frac{2}{3}\int_0^3 f(x).\cos\frac{2n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 (2x - x^2).\cos\frac{2n\pi x}{3}\,dx$	
	$= \frac{2}{3}\left[(2x - x^2)\frac{\left(\frac{\sin 2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)} - (2 - 2x)\frac{\left(\frac{-\cos n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + (-2)\left(\frac{\frac{-\sin n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right)\right]_0^3$	
	$= -\frac{9}{n^2\pi^2}$	1
	$b_n = \frac{1}{l}\int_0^{2l} f(x).\sin\frac{2n\pi x}{l}\,dx = \frac{2}{3}\int_0^3 (2x - x^2).\sin\frac{2n\pi x}{3}\,dx$	
	$= \frac{2}{3}\left[(2x - x^2)\left(\frac{-\frac{\cos 2n\pi x}{3}}{\frac{n\pi}{3}}\right) - (2 - 2x)\left(\frac{-\frac{\sin 2n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^2}\right) + (-2)\left(\frac{\frac{\cos 2n\pi x}{3}}{\left(\frac{n\pi}{3}\right)^3}\right)\right]_0^3$	1
	$= \frac{3}{n\pi}$	
	$\therefore f(x) = -\frac{9}{\pi^2}\sum \frac{\cos\frac{2n\pi x}{3}}{n^2} + \sum \frac{3}{n\pi}\sin\frac{2n\pi x}{3}.$	1
6.	$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$	1
	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2$	1

Sl No	Value Points / Answer	Mark
	$\therefore y_2 = \frac{-y_1^2}{y} = \frac{-4a^2}{y^3}$ using (1)	1
	$\rho = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{3/2}}{-\frac{4a^2}{y^3}} = \frac{\left[y^2+4a^2\right]^{3/2}}{-4a^2} = \frac{\left[4ax+4a^2\right]^{3/2}}{-4a^2}$ $\left[\because y^2=4ax\right]$	1
	$= \frac{(4a)^{3/2}\left[x+a\right]^{3/2}}{-4a^2} = -\frac{2}{\sqrt{a}}\left[x+a\right]^{3/2}$	1
	Taking the magnitude, $\rho = \frac{2}{\sqrt{a}}\left[x+a\right]^{3/2}$	
9.	Leaving aside the first term, $u_n = \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{n\left(1+\frac{1}{n}\right)^{n+1}}$	2
	Take $v_n = \frac{1}{n}$ $\underset{n\to\infty}{Lt} \frac{u_n}{v_n} = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \underset{n\to\infty}{Lt} \frac{1}{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}$	1
	$= \frac{1}{e}\times\frac{1}{1} = \frac{1}{e}$ which is finite & $\neq 0$	1
	$\Sigma v_n = \Sigma \frac{1}{n}$ diverges $\Rightarrow \Sigma u_n$ also diverges or converges together by Comparison test. since Σv_n diverges Σu_n also diverges	1
10.	Maclaurin's Series of $f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$	1
	$f(x) = \cos x$ $f(0) = 1$ $f'(x) = -\sin x$ $f'(0) = 0$ $f''(x) = -\cos x$ $f''(0) = -1$ $f'''(x) = \sin x$ $f'''(0) = 0$ $f^{iv}(x) = \cos x$ $f^{iv}(0) = 1$	4
	$\therefore f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	1
	$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$	

Sl.No	Value Points / Answer	Mark
	$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$	
	$= \frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$	2
	$= \frac{4(-1)^n}{n^2}$	
	$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$	1
7.	$\underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^{1/x^2}$ (1^{∞} form)	
	Let $y = \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right)^{1/x^2}$	
	$\log y = \underset{x\to 0}{Lt} \frac{1}{x^2}\log\left(\frac{\sin x}{x}\right)$	1
	$= \underset{x\to 0}{Lt} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$ ($\frac{0}{0}$ form), apply L'Hospital's rule	
	$= \underset{x\to 0}{Lt} \frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$	
	$= \underset{x\to 0}{Lt} \frac{\frac{x}{\sin x}\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$	2
	$= \left(\underset{x\to 0}{Lt}\frac{x}{\sin x}\right)\underset{x\to 0}{Lt}\left(\frac{x\cos x - \sin x}{2x^3}\right)$	
	$= 1 \times \underset{x\to 0}{Lt}\left(\frac{x\cos x - \sin x}{2x^3}\right)$ ($\frac{0}{0}$ form)	1
	$= \underset{x\to 0}{Lt}\frac{\cos x - x\sin x - \cos x}{6x^2} = \underset{x\to 0}{Lt}\left(\frac{-\sin x}{6x}\right)$	
	$= -\frac{1}{6} \times \underset{x\to 0}{Lt}\left(\frac{\sin x}{x}\right) = -\frac{1}{6}$	
	$\Rightarrow y = e^{-1/6}$	1
8.	Radius of Curvature, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$	1
	Given curve is $y^2 = 4ax$	
	Differentiating w.r.t x, $2yy_1 = 4a$	
	$y_1 = \frac{2a}{y} \longrightarrow (1)$	1
	Again differentiating,	
	$2y_1 \cdot y_1 + 2y y_2 = 0$	

Sl No.	Value Points / Answer	Mark

PART C

11(a) (i) Find the evolute of the curve $x^{2/3}+y^{2/3}=a^{2/3}$

Here $x=a\cos^3\theta$, $y=a\sin^3\theta$ — 1

$\frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$ — 1

$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\tan\theta\right)=-\frac{d}{d\theta}\left(\tan\theta\right)\frac{d\theta}{dx}$

$\therefore y_2=-\sec^2\theta\times\frac{1}{-3a\cos^2\theta\sin\theta}=\frac{1}{3a}\sec^4\theta\operatorname{cosec}\theta$ — 1

$X=x-\frac{y_1}{y_2}(1+y_1^2)=a\cos^3\theta-\frac{-\tan\theta\left(1+(-\tan\theta)^2\right)}{\frac{\sec^4\theta\operatorname{cosec}\theta}{3a}}$ — 1

$=a\cos^3\theta+3a\sin^2\theta\cos\theta \longrightarrow (1)$

$Y=y+\frac{1+y_1^2}{y_2}=a\sin^3\theta+\frac{(1+\tan^2\theta)}{\frac{\sec^4\theta\operatorname{cosec}\theta}{3a}}$ — 1

$=a\sin^3\theta+3a\sin\theta\cos^2\theta \longrightarrow (2)$ — 1

From (1) & (2) we have centre of curvature $C(X,Y)$ is given by $(X,Y)=\left(a\cos^3\theta+3a\sin^2\theta\cos\theta,\ a\sin^3\theta+3a\sin\theta\cos^2\theta\right)$

(1)+(2) $X+Y=a\left(\sin^3\theta+3\sin^2\theta\cos\theta+3\sin\theta\cos^2\theta+\cos^3\theta\right)$

$\Rightarrow X+Y=a(\sin\theta+\cos\theta)^3 \longrightarrow (3)$

(1)−(2) gives

$X-Y=a\left(-\sin^3\theta+3\sin^2\theta\cos\theta-3\sin\theta\cos^2\theta+\cos^3\theta\right)$

$\Rightarrow X-Y=-a(\sin\theta-\cos\theta)^3 \longrightarrow (4)$

Hence $(X+Y)^{2/3}+(X-Y)^{2/3}=\left\{\left[a(\sin\theta+\cos\theta)^3\right]^2\right\}^{1/3}+\left\{\left[-a(\sin\theta-\cos\theta)^3\right]^2\right\}^{1/3}$ — 1

$=a^{2/3}\left\{(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2\right\}$

$=2a^{2/3}\left(\sin^2\theta+\cos^2\theta\right)=2a^{2/3}$

$\therefore$ Evolute of (x,y) is $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$ — 1

Sl No	Value Points / Answer.	Mark

11(a)(ii) $f(x,y) = x^3+y^3-3axy$

$f_x = 3x^2-3ay \quad f_y = 3y^2-3ax$ — 2

$r = f_{xx} = 6x \quad s = f_{xy} = -3a \ , \ t = f_{yy} = 6y$ — 2

$f_x = 0 \Rightarrow 3x^2-3ay = 0$

$f_y = 0 \Rightarrow 3y^2-3ax = 0$

$\therefore x^2 = ay$. Hence $y = \dfrac{x^2}{a}$

$\therefore \dfrac{3x^4}{a^2} - 3ax = 0 \Rightarrow x = 0$ or $x = a$ — 1

Hence the stationary points are $(0,0)$ and (a,a)

Now $rt-s^2 = 36xy - 9a^2$ — 1

At $(0,0)$ $rt-s^2 < 0$

$\therefore$ At $(0,0)$ the function is neither maximum nor minimum.

$\therefore (0,0)$ is a saddle point

At (a,a) — 1

$rt-s^2 = 36a^2-9a^2 = 27a^2 > 0$

Now if a is positive, then r is positive.

if a is negative, then r is negative.

Hence (a,a) gives maximum if $a<0$ and minimum if $a>0$ — 1

11(b)(i) $f(x,y) = \sin x + \sin y + \sin(x+y)$ — 1

$f_x = \cos x + \cos(x+y) \ , \ f_y = \cos y + \cos(x+y)$

$r = f_{xx} = -\sin x - \sin(x+y), \ f_y = \cos y + \cos(x+y)$

$s = f_{xy} = -\sin(x+y)$

$t = f_{yy} = -\sin y - \sin(x+y)$ — 2

$f_x = 0$ & $f_y = 0$

$\Rightarrow \cos x + \cos(x+y) = 0 \longrightarrow (1) \qquad \cos y + \cos(x+y) = 0 \longrightarrow (2)$

$(1)-(2) \Rightarrow \cos x - \cos y = 0$

$\Rightarrow \cos x = \cos y$ — 1

$\Rightarrow x = y$

From (1) $\cos x + \cos 2x = 0$ or ~~$\cos x - \cos y = 0$~~

or $\cos 2x = -\cos x = \cos(\pi - x)$ or $2x = \pi - x$ — 1

$\therefore x = \pi/3 \quad \therefore x = y = \dfrac{\pi}{3}$ is a stationary point.

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $r = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3} < 0$, $s = \dfrac{\sqrt{3}}{2}$, $t = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3}$

$rt - s^2 = 3 - \dfrac{3}{4} = \dfrac{9}{4} > 0$. — 1

Sl NO	Value Points / Answer	Mark
	Also $r<0$	
	$\therefore f(x,y)$ has a maximum value at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$	1
	Maximum value $= f\left(\frac{\pi}{3},\frac{\pi}{3}\right) = \sin\frac{\pi}{3}+\sin\frac{\pi}{3}+\sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$	1
11(b)(ii)	Here z is a composite function of u and v	
	$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^{u}+\frac{\partial z}{\partial y}(-e^{-u})$	3
	$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})$	3
	$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v} = (e^{u}+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^{v})\frac{\partial z}{\partial y} = x\cdot\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$	1
12(a)(i)	$u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$, $u_{n+1} = \frac{x^{2n}}{(n+2)(\sqrt{n+1})}$	2
	$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}}\times\frac{1}{x^2} = \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\sqrt{1+\frac{1}{n}}\cdot\frac{1}{x^2}$	1
	$\underset{n\to\infty}{Lt}\ \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$	
	By D'Alembert's Ratio Test, Σu_n Converges if $\frac{1}{x^2}>1 \Rightarrow x^2<1$	2
	and diverges if $\frac{1}{x^2}<1 \Rightarrow x^2>1$	
	When $x^2=1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)}$	2
	Take $v_n = \frac{1}{n^{3/2}}$	
	$\Sigma v_n = \Sigma\frac{1}{n^{3/2}}$ Converges by Comparison test ($\because p>1$)	
	$\Rightarrow \Sigma u_n$ also Converges.	
	$\therefore \Sigma u_n$ is Convergent if $x^2\le 1$ and divergent if $x^2>1$	1
12(a)(ii)	Given $y = \frac{1}{\sqrt{1+2x}}$	
	$\Rightarrow y^2(1+2x)=1$	1
	Differentiating w.r.t x	
	$2yy_1(1+2x)+2y^2=0$	2
	$\Rightarrow y_1(1+2x)+y=0$	
	Taking n^{th} derivative,	1
	$D^n[y_1(1+2x)]+D^n y=0$	

Sl No	Value Points / Answer	Mark
	Using Leibnitz formula	
	$(1+2x)D^n y_1 + nC_1 D(1+2x) D^{n-1} y_1 + nC_2 D^2(1+2x) D^{n-2} y_1 + D^n y = 0$	1
	$(1+2x) y_{n+1} + n(2) y_n + 0 + y_n = 0$	1
	$\Rightarrow (1+2x) y_{n+1} + (2n+1) y_n = 0$	1
12(b)(i)	The given Series is $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{\sqrt{n}}$	1
	$\lvert u_n \rvert = \frac{\lvert x^n \rvert}{\sqrt{n}} = \frac{\lvert x\rvert^n}{\sqrt{n}}$	
	$\lvert u_{n+1} \rvert = \frac{\lvert x \rvert^{n+1}}{\sqrt{n+1}}$	2
	$\frac{\lvert u_n\rvert}{\lvert u_{n+1}\rvert} = \sqrt{\frac{n+1}{n}} \times \frac{1}{\lvert x\rvert} = \sqrt{1+\frac{1}{n}} \times \frac{1}{\lvert x\rvert}$	
	$\lim_{n\to\infty} \frac{\lvert u_n\rvert}{\lvert u_{n+1}\rvert} = \frac{1}{\lvert x\rvert}$	1
	$\therefore$ By Ratio test, the series $\sum \lvert u_n\rvert$ is Convergent if $\frac{1}{\lvert x\rvert} > 1$ ie if $\lvert x\rvert < 1$ ie; if $-1 < x < 1$ and divergent if $\frac{1}{\lvert x\rvert} < 1$. ie; if $\lvert x\rvert > 1$ ie if $x > 1$ or $x < -1$	2
	Ratio test fails when $\lvert x\rvert = 1$, ie; when $x = 1$ or $x = -1$	
	When $x = 1$, the Series becomes $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ which is an alternating Series and is Convergent.	1
	When $x = -1$, the series becomes $-\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots\right) = -\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent by p-series test.	
	Hence the given series converges for $-1 < x \leq 1$	1
12(b)(ii)	Given $y = a\cos(\log x) + b\sin(\log x)$	
	Differentiating both sides w.r.t x	
	$y_1 = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$	1
	$x y_1 = -a\sin(\log x) + b\cos(\log x)$	
	Again diff. w.r.t x,	
	$x y_2 + y_1 = -a\cos(\log x) \times \frac{1}{x} - b\sin(\log x) \times \frac{1}{x}$	
	$\Rightarrow x^2 y_2 + x y_1 = -y$	2

Sl. No	Value Points / Answer	Mark
	$\therefore D^n(x^2y_2) + D^n(xy_1) + D^n(y) = 0$	1
	using Leibnitz formula.	
	$x^2 y_{n+2} + \left[n(2x) y_{n+1} + \frac{n(n-1)}{1.2}(2) y_n\right] + \left[x y_{n+1} + n(1) y_n\right] + y_n = 0$	2
	$\Rightarrow x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$	1
13(a)(i)	Let $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$	
	Char. eqn is $\lvert A - \lambda I \rvert = 0 = \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$	1
	$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$ $\Rightarrow \lambda = -1$ and $\lambda = 6$	4
	Hence $\frac{1}{6}$ and -1 are the eigen values of A^{-1}.	1
	$\therefore A^{-1} = \frac{1}{6}\begin{bmatrix} -4 & -2 \\ -5 & -1 \end{bmatrix}$	1
13(a)(ii)	Let the given system of equations by $AX = B$.	
	$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$	1
	Augmented matrix. $[A:B] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$	
	$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix}$ $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$	1
	$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$ $R_2 \to R_2/-7$, $R_3 \to R_3 + 6R_2$, $R_4 \to R_4 + 3R_2$	1
	$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix}$ $R_3 \to \frac{7}{5} R_3$	1

Sl No	Value Points / Answer	Mark
	$[A:B] \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 \to R_4 + \frac{R_3}{7}$ which is an echelon form.	
	Now $AX = B$ becomes $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/7 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8/7 \\ 4 \\ 0 \end{bmatrix}$	2
	By Back substitution, $z = 4$ $y - \frac{5}{7}z = \frac{8}{7}$ $y = \frac{8}{7} + \frac{5}{7} \times 4 = 4$	1
	$x + 2y - z = 3$ $x = 3 - 2y + z$ $= 3 - 8 + 4 = -1$	
	$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$	1
13(b)	The matrix of the quadratic form is $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$	2
	The characteristic roots of A are given by $\lvert A - \lambda I \rvert = 0$ ie; $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$	3
	$\therefore \lambda = 0, 3, 15$	1
	Characteristic vector for $\lambda = 0$ is given by $[A - (0)I]X = 0$	1
	$8x_1 - 6x_2 + 2x_3 = 0$ $-6x_1 + 7x_2 - 4x_3 = 0$ $2x_1 - 4x_2 + 3x_3 = 0$ solving first two, we get $\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$ $\therefore X_1 = k_1 [1, 2, 2]'$	2
	When $\lambda = 3$, the characteristic vector is given by $[A - 3I]X = 0$	

Sl No	Value Points / Answer	Mark
	$5x_1 - 6x_2 + 2x_3 = 0$ $-6x_1 + 4x_2 - 4x_3 = 0$ $2x_1 - 4x_2 = 0$	
	Solving any 2 equations, we get $X_2 = K_2[2, 1, -2]'$	2
	$\|\|^{ly}$ characteristic vector corresponding to $\lambda = 15$ is $X_3 = K_3[2, -2, 1]'$	1
	Now X_1, X_2, X_3 are pairwise orthogonal. $X_1 \cdot X_2 = X_2 \cdot X_3 = X_3 \cdot X_1 = 0$	1
	$\therefore$ the normalised modal matrix is $B = \left[\frac{X_1}{\|\|X_1\|\|}, \frac{X_2}{\|\|X_2\|\|}, \frac{X_3}{\|\|X_3\|\|}\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$	1
	Now B is orthogonal matrix and $\|B\| = 1$ $\therefore B^{-1} = B^T$ & $B^{-1}AB = D = \text{diag}(0, 3, 15)$	
	ie $\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} A \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$	
	$X'AX = Y'(B^{-1}AB)Y = Y'DY$ $= [y_1, y_2, y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 3y_2^2 + 15 \cdot y_3^2$	1
	which is the required canonical form.	
14(a)(i)	Since $\sin ax$ is an odd function of x, $a_0 = 0$ and $a_n = 0$	
	Let $\sin ax = \sum_{n=1}^{\infty} b_n \sin nx$	1
	$\Rightarrow b_n = \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx\, dx$	1
	$= \frac{1}{\pi}\int_0^{\pi}\left[\cos(n-a)x - \cos(n+a)x\right]dx$	
	$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{(n-a)} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}$	1

Sl No	Value Points /Answer	Mark
	$= \frac{1}{\pi}\left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right] = \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right]$	2
	$= \frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{n-a} + \frac{1}{n+a}\right]$	
	$= (-1)^{n+1} \cdot \frac{2n \sin a\pi}{\pi(n^2-a^2)}$	2
	$\therefore \sin ax = \frac{2\sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2} \sin nx.$	
	$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \dots\right]$	1
14(a)(ii)	$e^x = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad (\text{since } l=1)$	1
	Then $b_n = 2\int_0^1 e^x \sin n\pi x \, dx$	1
	$= 2\left[+\frac{e^x}{1+(n\pi)^2}\left(\sin n\pi x - n\pi \cos n\pi x\right)\right]_0^1$	1
	$= 2\left[\frac{e}{1+(n\pi)^2}(-n\pi \cos n\pi) - \frac{1}{1+(n\pi)^2}(-n\pi)\right]$	
	$= \frac{2}{1+n^2\pi^2}\left[-en\pi(-1)^n + n\pi\right]$	
	$= \frac{2n\pi}{1+n^2\pi^2}\left[1 - e(-1)^n\right]$	3
	$\therefore e^x = 2\pi \sum_{n=1}^{\infty} \frac{n\left[1-e(-1)^n\right]}{1+n^2\pi^2}$	
	$= 2\pi\left[\frac{1+e}{1+\pi^2}\sin \pi x + \frac{2(1-e)}{1+4\pi^2}\sin 2\pi x + \frac{3(1+e)}{1+9\pi^2}\sin 3\pi x + \dots\right]$	1
14(b)	Here length of the interval is π, ie $l = \pi/2$	
	$y = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{\pi/2} + b_1 \sin\frac{\pi x}{\pi/2}\right) + \left(a_2 \cos\frac{2\pi x}{\pi/2} + b_2 \sin\frac{2\pi x}{\pi/2}\right) + \dots$	
	$= \frac{a_0}{2} + (a_1 \cos 2x + b_1 \sin 2x) + (a_2 \cos 2x) + b_2 \sin 4x) + \dots$	2
	The values of. x, y, $\sin 2x$, $\cos 2x$, $\sin 4x$, $\cos 4x$ are tabulated below.	

Sl No	Value Points / Answer	Mark

x	y	$\cos 2x$	$\sin 2x$	$\cos 4x$	$\sin 4x$
0	0	1	0	1	0
$\frac{\pi}{6}$	9.2	0.5	0.87	−0.5	0.87
$\frac{2\pi}{6}$	14.4	−0.5	0.87	−0.5	−0.87
$\frac{3\pi}{6}$	17.8	−1	0	1	0
$\frac{4\pi}{6}$	17.3	−0.5	−0.87	−0.5	0.87
$\frac{5\pi}{6}$	11.7	0.5	−0.87	−0.5	−0.87

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$$a_0 = \frac{2}{l}\Sigma y = \frac{2}{6}(70.4) = 23.46$$ 1

$$a_1 = \frac{2}{l}\Sigma y\cos 2x = \frac{2}{6}\left[0.5\,(9.2-14.4-17.3+11.7)-17.8\right]$$ 1

$$= \frac{1}{3}\left(-23.2 = -7.73\right.$$

$$a_2 = \frac{2}{l}\Sigma y\cos 4x = \frac{2}{6}\left[0.5(9.2+14.4+17.3+11.7)+17.8\right]$$ 1

$$= \frac{1}{3}\left[-26.3+17.8\right] = \frac{1}{3}\left[-8.5\right]$$

$$= -2.83$$

$$b_1 = \frac{2}{l}\Sigma y\sin 2x = \frac{2}{6}\left[0.87\,(9.2+14.4-17.3-11.7)\right]$$ 1

$$= \frac{1}{3}\left[0.87\,(-5.4)\right] = -1.566$$

$$b_2 = \frac{2}{l}\Sigma y\sin 4x = \frac{2}{6}\left[0.87\,(9.2-14.4+17.3-11.7)\right]$$ 1

$$= \frac{1}{3}\left[0.87\,(4)\right] = 0.116$$

$$\therefore y = 11.73 - (7.73\cos 2x + 1.566\sin 2x) + (-2.83\cos 4x + 0.116\sin 4x)$$ 1

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